ON GENERAL TOPOLOGY AND THE RELATION OF THE PROPERTIES OF THE CLASS OF ALL CONTINUOUS FUNCTIONS TO THE PROPERTIES OF SPACE*

BY

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INTRODUCTION

Successive generalizations of the theory of sets of points based in turn on the concepts distance, limit of a sequence, neighborhood, and limiting point or point of accumulation led to the recognition of the concept point of accumulation as fundamental for the general theory of abstract sets.† The first writer formally to suggest founding the theory of abstract sets on this basis was F. Riesz‡ who proposed a set of postulates characteristic of a very general type of abstract set. The existence of a definition of point of accumulation for the subsets E of an aggregate P is logically equivalent to the existence of a single-valued function

$$E' = K(E).$$

whose argument E ranges over the class $\mathfrak{S}$ of all subsets E of the class P and whose values are likewise elements of $\mathfrak{S}$. Thus the theory of abstract sets is a phase of the study of *set-valued set-functions*, in fact of those functions which transform an aggregate $\mathfrak{S}$ of sets of elements of a class P into an aggregate $\mathfrak{S}'$ of the same character.

Since analysis situs or topology is primarily concerned with invariants under homeomorphic transformations of a space into itself the general theory of abstract sets provides a basis for the most general treatment of topological problems. For this reason F. Hausdorff§ called a very general

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type of abstract set a topological space. For the same reason Fréchet*
has proposed that the term *topological space* refer to the most general form of
abstract set, the case in which the points of accumulation are defined by an
arbitrary set-function. This terminology will be used throughout the
following discussion. A topological space of the Hausdorff type will be
referred to as a Hausdorff space.

The first part of the present paper may be regarded as a discussion of the
following problem: Determine the extent of the general theory of topological
spaces.† It is also a study of various set-valued set-functions which may be
defined in terms of a postulated arbitrary set-function $K(E)$. By the aid of
suitable definitions many important theorems of the theory of sets of points
are extended to the most general topological space.

Several of the set-functions defined by a given set-valued set-function
$K(E)$ are of particular interest. The function $V(E)$ of §9 is associated with
the concept neighborhood and defines a neighborhood space associated with
the given topological space. The class of *proper nuclear* points‡ of a set E
defines a set-function discussed in §11. The importance of this set-function
was first recognized by C. Kuratowski and W. Sierpinski§ who used it in the
solution of the following problem of Fréchet: *Determine the most general
class* (L) *in which the theorem of Borel is true.*

It is of interest to observe that while the Hedrick property,‖ *the derived
set of every set is closed,* may not be present in a topological space, it is
present in associated spaces discussed in §§8 and 12. This fact permits the
generalization to general topological spaces of many classical theorems.

The second part of the paper is devoted to a study of covering theorems
in general topological space.¶ Of particular interest is the formulation in §20
of two sets of necessary and sufficient conditions that a set possess the
"any to finite" form of the property of Borel.

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* Congrès des Sociétés Savantes, Dijon, 1924, and Comptes Rendus, vol. 180 (1925), p. 419.
See IV, p. 166.

† The results presented in this part of the paper were obtained prior to the appearance and
independently of the article *La notion de dérivée comme base d'une théorie des ensembles abstraits,*
by W. Sierpinski, Mathematische Annalen, vol. 97 (1926), pp. 321–337. An examination of the work
of Sierpinski shows that his article duplicates the results here presented in part only.

‡ In a Hausdorff space these points coincide with the *complete limiting* points of P. Alexandroff
and P. Urysohn, Mathematische Annalen, vol. 92 (1924), p. 258, and in general topological space
with the *complete interior limiting points* of T. H. Hildebrandt, Bulletin of the American Mathematical
Society, vol. 32 (1926), p. 469.

§ Fundamenta Mathematicae, vol. 2 (1921), pp. 172–8.

‖ E. R. Hedrick, these Transactions, vol. 12 (1911), pp. 285–294.

¶ T. H. Hildebrandt has given (loc. cit) an excellent résumé of the literature on covering theorems
entitled *The Borel theorem and its generalizations.*

In the third part of this paper we consider the relationship between the properties of the class C of all continuous functions on a topological space and the properties of the space. It is usual in the theory of abstract sets for writers to impose conditions on the space and deduce the properties of the class C which result. It is evident, however, that many properties of the class C cannot be present in spaces of unrestricted generality. For example, if the space is enumerable and connected every continuous function is constant. Thus the assumption of the existence of a non-constant function restricts the character of the space. Fréchet has proposed, in correspondence with P. Urysohn and myself, the following problem: Characterize the most general space in which there exists a non-constant continuous function.

The solution of this problem presented in §21 is a generalization of a result of P. Urysohn* stated for Hausdorff spaces.

A. D. Pitcher† has shown for the space of a symmetric écart‡ that if the class C of all continuous functions is *uniformly proper*§ the space is compact, connected, and metric. In this paper we consider a group of fundamental properties possessed by the class C of all continuous functions on a metric space, and obtain a number of topological conditions which are necessary for the presence of combinations of these properties in the case of a topological space. These results extend and supplement those obtained by Pitcher.

It is important to note that the definition of continuous function on a topological space given by Fréchet‖ is in fact a neighborhood definition, and that in consequence the theory of continuous functions on the classes (V) is as general as that obtained in the case of the most general topological space.

I. DERIVED SET-FUNCTIONS

1. **Set-functions.** Throughout the following discussion the term *space* or *topological space* will be used to denote a system (P, K) composed of an

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* Mathematische Annalen, vol. 94 (1925), p. 290.

† These Transactions, vol. 19 (1918), pp. 73–76.

‡ A symmetric non-negative function of pairs of points p, q, which is defined for all pairs of points of the class P, and vanishes if and only if $p=q$. Cf. Fréchet, these Transactions, vol. 19 (1918), p. 53–65.

§ A class of continuous functions is uniformly proper if every function of the class is bounded, uniformly continuous, assumes every value between its bounds, and has the following further properties: if p and q are distinct points there is a function $f(p)$ of the class such that $f(p) \neq f(q)$; if A and B are any two sets such that the écart of pairs of points one from each set has a positive minimum, there is a continuous function $f(p)$ such that $f(p) = 0$, if p is in A, $=1$, if p is in B, and $0 \leq f(p) \leq 1$, for all other points.

‖ Loc. cit. III, p. 363.

abstract aggregate or set P and a relation $E'KE$ between subsets E, E' of P.

The set E' is assumed to be unique and determined for each subset E of the fundamental set P. Thus the relation $E' = K(E)$ defines a single-valued set-valued set-function, whose range is the class of all subsets of P, and whose values are also subsets of P. Different set-functions $K(E)$, $H(E)$, relative to the same set P determine different spaces.

2. **Properties of sets of points.** The elements of the class P in a space (P, K) will be called points. The elements, if any, of the set $E' = K(E)$ will be called *points of accumulation* of the set E with reference to the function $K(E)$, relation K, or space (P, K). The terms of the theory of sets of points retain their usual significance in relation to the points of accumulation defined by a set-function.

In case several set-functions are to be considered simultaneously we indicate the function of reference by writing K-point, K-closed, K-interior, etc.

3. **Derived set-functions.** In this and following paragraphs we consider a number of set-functions which may be defined in terms of a given set function $K(E)$, together with the corresponding spaces and types of points of accumulation.

The complement $P - E$ of a set E may be represented by the set function $C(E)$. In this notation the definition of interiority may be stated as follows. *A point p is K-interior to a set E if it is not a K-point of any subset of $C(E)$.* The set $I(E)$ of all the K-interior points of a set E is of great importance.

The function $I(E)$ is easily seen to be monotonic increasing. That is, if E is a subset of a set G, $I(E) \leq I(G)$.

Open sets are defined by the equation $E = I(E)$, that is, they consist entirely of interior points. It is quite easy to show that the sum of a finite or infinite number of open sets is open. The corresponding proposition regarding the product of closed sets cannot be extended to general space.

The frontier of a set E, $F(E)$, is given by the formula

$$F(E) = E \cdot K[C(E)] + C(E) \cdot K(E).$$

The isolated points of a set E define the function $S(E)$.

4. **Iteration of set-functions.** By finite iteration we obtain from a given set-function $K(E)$ the derived sets and derived functions of finite order

$$E^{(\alpha)} = K^{(\alpha)}(E).$$

Then for ordinals of the second kind we define

$$E^{(\alpha)} = K^{(\alpha)}(E)$$

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* M. Fréchet, loc. cit. II.

as the outer limit of $E^{(\beta)}$ where β varies over the set of all ordinals preceding α. Finally we set

$$E^{(\alpha+1)} = K^{(\alpha+1)}(E) = K\left[K^{(\alpha)}(E)\right].$$

Since the function K is general we have defined the iterates of finite and transfinite order for all set-valued set-functions.

A set-function may be cyclic under iteration. For example, the function $C(E) = P - E$ is cyclic of period two.

5. **Composition of set-functions.** From two set-functions G, H we may derive others by various forms of composition. For example, the sum $G+H$ and the product GH are defined by the equations

$$\overline{G + H}(E) = G(E) + H(E), \quad GH(E) = G(E) \cdot H(E).$$

We may also consider functions of functions, as $G[H(E)]$.

6. **Fundamental properties of set-functions.** For convenience of reference we present a list of the properties of set-functions which are fundamental for the present study.*

I. (Monotonic.) *If a set A is a subset of a set B, then $K(A)$ is a subset of $K(B)$.*

II. *If $E = A + B$, then $K(E) \leqq K(A) + K(B)$.*

III. *For every set E, $K''(E) = (K'E)$.*†

IV. *A set of one element has no K-point.*

V. *Each point of a set E is uniquely determined by the family of subsets of E of which it is a K-point.*

If a set-function has both of the properties I and II it is *additive*, that is

$$K(A + B) = K(A) + K(B).$$

It follows readily from the third property (III) that all the iterates of the function $K(E)$ have the property, and furthermore, that for any two ordinal numbers α, β ($\alpha < \beta$),

$$K^{(\alpha)}(E) \leqq K^{(\beta)}(E).$$

Properties I and IV together imply that the null set has no K-point.

7. **Monotonic set-functions.** From every set-function $K(E)$ we derive a monotonic set-function $L(E)$ of considerable interest. The L-points of a

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* Of these properties, I, II, IV, V are equivalent to the four properties of F. Riesz, loc. cit. The third corresponds to the Hedrick property (loc. cit.).

† A space with this property is said to be *accessible*. Cf. Fréchet, loc. cit. I.

set E are defined to be those points of the space (P, K) which are K-points of some subset of E. Obviously

$$K(E) \leqq L(E).$$

It is also evident that the function $L(E)$ is monotonic. *If the function $K(E)$ is monotonic, $K(E) = L(E)$ for every set E.*

The terms K-interior and L-interior are equivalent. Hence every open set in the space (P, K) is open in the derived space (P, L), and vice versa.

The property L-closed is stronger than the property K-closed. The complement of an open set in the space (P, K) is L-closed, and the complement of an L-closed set is open in both spaces. Because of these facts we shall in considering general topological spaces refer to the sets which are L-closed as *completely* closed. The notation F is reserved for the completely closed sets, preserving the usual formulas,

$$C(O) = F, \quad C(F) = O.$$

Since the sum of a family of open sets is an open set it follows at once that *the product of a finite or infinite family of completely closed sets in a general topological space is completely closed.*

8. A derived set-function with the Hedrick property. Let $M^0(E)$ denote the set common to all completely closed sets containing the set E as subset. This set exists (since the set P is completely closed) and is completely closed. The function $M^0(E)$ has the fundamental properties I and III. It will now be shown that this function may be defined in terms of the function $L(E)$ and the process of iteration.

Let a set-function $M(E)$ be defined by the equation

$$M(E) = E + L(E).$$

By iteration we obtain

$$M''(E) = M(E) + LM(E).$$

The series of functions $M^{(\alpha)}(E)$ derived from $M(E)$ by iteration has the property that if α precedes β then $M^{(\alpha)}(E)$ is a subset of $M^{(\beta)}(E)$. If we accept the theory of well ordered series in general, there must exist an ordinal for which

$$M^{(\alpha+1)}(E) = M^{(\alpha)}(E).$$

The set $M^0(E) = M^{(\alpha)}(E)$ so defined is the least completely closed set containing E. Evidently the iterates of the function $M^0(E)$ are identical to M^0.

But $M^0(E)$ will in general contain points of E which are not points of accumulation in the usual sense. However we may say that a point p is a

point of accumulation of a set E of *degree* α provided α is the least ordinal for which the point p is an L-point of $M^{(\alpha)}$ (E). The set of all points which are points of accumulation of E of degree α for some value of α will be denoted by $L^0(E)$. This set is also completely closed. We have thus obtained the closure of derived sets in another manner.

THEOREM. *Every topological space* (P, K) *determines an accessible space* (P, L^0). *The spaces coincide if the function* $K(E)$ *has properties* I *and* II. *For all values of the ordinal* α

$$K^{(\alpha)}(E) \leqq L^0(E).$$

If a set E is closed or open in the space (P, L^0) it is completely closed, or open, in any of the spaces (P, K), (P, L), $(P, K^{(\alpha)})$, $\alpha = 1, 2, 3, \cdots$.

The significance of this theorem in the case of certain singular set-functions is illustrated by the example $K(E) = C(E)$. Then $L^0(E) = P$.

9. **Neighborhood spaces.** Any set V of points of a space (P, K) which has a point p for an interior point will be called a *neighborhood* of p. Then the set function $V(E)$ may be defined by the set of all points p such that every neighborhood of p contains a point of E distinct from p. The function $V(E)$ is monotonic increasing and has the further property

IIa. *If a point p is a V-point or neighborhood point of a set E, it is a neighborhood point of the set $E - p$.*

The space (P, V) so defined is a class (V) of Fréchet.* It has been shown by Fréchet that properties I and IIa are together a necessary and sufficient condition that a function $K(E)$ be identical to the derived function $V(E)$.

Every neighborhood point of a set E is an L-point. If a point p is a K-point of the set $E - p$ it is a neighborhood point of E. If a point p is a K-point of a set E, but is not a K-point of the set $E - p$ or any subset of that set, it is not a V-point of E. The K-points which are not neighborhood points are virtually isolated, and will be called *singular* points of accumulation.

The neighborhoods derived from the function $V(E)$ are identical to those derived from the function $K(E)$.

If the null set has no L-point, then each point has one neighborhood at least (which may be the aggregate P) and every neighborhood of a point p contains p. This is property A of the Hausdorff postulates for a neighborhood space.†

If in the definition of the function $V(E)$ we replace the family of all

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* Loc. cit. I.
† Loc. cit., p. 213.

neighborhoods of a point p by the family of all open sets O which contain p we obtain a function $O(E)$. Clearly $V(E) = O(E)$. The function $O(E)$ has properties I, IIa, and III. A necessary and sufficient condition that the functions $V(E)$ and $O(E)$ be identical is that for each point p there exist a neighborhood V such that $M^0(P-V)$ does not contain the point p. It is quite easy to show that this is the case if the function $K(E)$ has the properties I, II, and III.

10. Nuclear points. Hausdorff* has introduced the class E_μ of all points p of order μ at least relative to E; that is, the class of all points such that every neighborhood contains a subset of E of power μ. It is evident that if $\mu < \nu$ then $E_\mu \geq E_\nu$. This definition defines a series of functions denoted by $V_\mu(E)$ ($\mu = 1, 2, 3, \cdots$). A point of order μ will be called a μ-point. If it is necessary for the sake of clearness, we may write K_μ-point, L_μ-point etc.

If a set E contains the set $V_\mu(E)$ it is μ-closed.

Since each neighborhood of a point p contains p we have at once

$$E + K_1(E) = K_1(E) = L_1(E) = V_1(E).$$

If $\mu = 1$, every μ-point is an L-point. The case $\mu = 0$ is trivial.

The μ-points of a set E of order $\mu = |E|$, the cardinal number of E, are called *nuclear*† points since they are V-points of the nucleus of E, the set of all points of E which are of order $|E|$ relative to E. Every nuclear point of a set of more than one element is an L-point.

If every infinite subset of a set E has a nuclear point in E, the set E is said to be self-nuclear. *Every self-nuclear set is self-compact relative to the function $L(E)$.*

11. Proper points of accumulation. A point p is a *proper* point of accumulation of a set E if for every neighborhood V of p there is a point q of E interior to V.‡ Every proper point of accumulation is a V-point. A K-point which is not a proper point of accumulation is said to be *improper*.

If the system of neighborhoods is equivalent to a family of open sets every point of accumulation is proper.

Among the proper points of accumulation are the points of *power μ*§ which are related in an interesting and remarkable manner to the covering theorems.

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* Loc. cit., p. 219 ff.

† E. W. Chittenden, Bulletin of the American Mathematical Society, vol. 30 (1924), p. 514. They correspond to the complete points of accumulation of Alexandroff and Urysohn in the case of a Hausdorff space (loc. cit.).

‡ Cf. T. H. Hildebrandt, loc. cit.

§ The terms "order" and "power" are due to Hildebrandt, loc. cit.

A point p is of *power μ* at least relative to a set E if every neighborhood of p contains a subset of E of power μ in its *interior*.

If $\mu > 0$, the points of power μ are also of order μ, and may therefore be called proper μ-points. The class of all proper μ-points of a set E is denoted by $H_\mu(E)$. A proper V-point is therefore an H_1-point.

The points of power $|E|$ relative to E are called *proper nuclear* points.[*] If a set E contains a proper nuclear point of every infinite subset of E, E is said to be *properly self-nuclear*.

If every infinite subset of a set E determines a proper K-point in E, E is *properly self-compact.*[†] In a space (P, L) every properly self-nuclear set E is properly self-compact.

A set E is *properly closed* if it contains all its proper points of accumulation. This closure is understood to be relative to the type of point of accumulation under consideration.

THEOREM. *The class $H_2(E)$ is properly closed in the space (P, L).*

A point p is a proper interior point of a set E in case every set A which has p for a V-point contains an ordinary interior point of E which is distinct from p. Let $D(E)$ denote the set of all proper interior points of E. The points of $I(E) - D(E)$, if any, are K-points of some subset of $P - I(E)$, but not of $P - E$. They may however be points of accumulation of the set $C(E)$ of some degree greater than the first.

We note the following evident propositions:

Every point of an open set is a proper interior point.

The set $D(E)$ is interior to $I(E)$.

In the space (P, H_1) the set $I(E)$ is open.

12. **Other set-functions.** In case μ is equal to the cardinal number of the positive integers the function $H_\mu(E)$ has the first four of the fundamental properties listed in §6. The corresponding space is therefore a class (H) of Fréchet.[‡] For this reason we denote the set $H_\mu(E)$, when $\mu = \aleph_0$, by $H(E)$. It follows that all the theorems which have been established for classes (H) may be extended when suitably modified to general topological space.

Another set-function of this kind determines an accessible space. A point p will be called a J-point of a set E if the interior of each neighborhood of p contains at least two points of E.

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[*] In the article cited above I called this type of point hypernuclear.

[†] On page 195 (loc. cit. IV) Fréchet introduces the concept compact in "sens très strict" which is equivalent for classes (V) to properly compact. Many of the theorems stated by Fréchet are extended in following sections to general topological space.

[‡] *Esquisse*, p. 354 and p. 366.

It follows at once that $J(E) = V(E)$, that is, that every J-point is a V-point. But the space (P, J) does not in general satisfy the condition IIa, and so need not be a neighborhood space. However, the function $J(E)$ has the fundamental properties I, III, and IV. But a function $K(E)$ with these properties is not necessarily equivalent to the corresponding derived function $J(E)$. The following proposition which is an immediate consequence of the definition of the function $J(E)$ is of interest.

The interior $I(E)$ of a set E in a general topological space is an open set in the derived space (P, J).

II. COVERING THEOREMS IN GENERAL TOPOLOGICAL SPACE

13. Properties of coverings. A family $\mathfrak{F}$ of sets G in a space (P, K) is a covering of a set E if every point p of E is an element of some set G of $\mathfrak{F}$; a proper covering in case each point p is *interior* to some set G. More generally a covering $\mathfrak{F}$ is of type T if each point p is in a relation T to some set G of $\mathfrak{F}$. From the axiom of choice we may conclude that every covering $\mathfrak{F}$ contains a covering $\mathfrak{F}_1$ of the same type and of cardinal number $|E|$ at most.

A covering $\mathfrak{F}$ of a set E is *reducible* if it contains a covering of the same type and of lower cardinal number, otherwise it is irreducible.

A covering $\mathfrak{F}$ of type T and power μ is *normal* if it can be arranged in a normally ordered series

$$G_1, G_2, G_3, \cdots, G_\alpha, \cdots \qquad (\alpha < \Omega_\mu),$$

where Ω_μ is the first ordinal of power μ_r and there is for each of the ordinals α a point q_α of E in the relation T to G_α, which is not in that relation to any G_β, where $\beta < \alpha$. The set $Q = [q_\alpha | \alpha < \Omega_\mu]$ will be said to be *associated* with the normal covering $\mathfrak{F}$. Every normal covering is irreducible. It is evident that *the set of points of Q in the relation T to G_α is of power less than μ for every $\alpha < \Omega_\mu$.*

THEOREM. *Every covering $\mathfrak{F}$ of a set E contains a normal covering of E of the same type.*

Let $\mathfrak{F}_1$ be any irreducible subfamily of $\mathfrak{F}$ which covers a given set E and let

(1) $$G_1, G_2, G_3, \cdots, G_\alpha, \cdots \qquad (\alpha < \Omega_\mu)$$

represent any arrangement of the sets G of the family $\mathfrak{F}_1$ in a well ordered series of type Ω_α. Let α_1 be the first of the indices α for which there is a point p_1 of E in the relation T to G_α. In general G_{α_β} contains a point p_β of E not in the relation T to G_{α_γ} if $\gamma < \beta$. The index β must range over an ordinal series

coterminal with that of α, therefore ordinally similar to that series, since the family $\mathfrak{F}_1$ is irreducible. The family

$$\mathfrak{F}_2 = [G_{\alpha_\beta} \,|\, \beta < \Omega_\mu]$$

is easily seen to be a normal covering of E of type T and power μ.

14. **The theorem of Borel in a Hausdorff space.** P. Alexandroff and P. Urysohn* have formulated for Hausdorff spaces the following elegant generalizations of the classical theorem of Borel.

THEOREM I_0. *In a Hausdorff space the following properties of a set E are equivalent.*

A_0. *Every enumerable subset of E has a nuclear point.*

B_0. *The product of every enumerable decreasing sequence of non-null closed subsets of E contains a point of E.*

C_0. *Every enumerably infinite family of open sets which covers E is reducible.*

A necessary and sufficient condition that a set E possess these three properties is that it be self-compact.

THEOREM II. *In a Hausdorff space the following properties of a set E are equivalent.*

A. *Every infinite subset of E has a nuclear point in E.*

B. *The product of every well ordered decreasing sequence of non-null closed subsets of E contains a point of E.*

C. *Every infinite family of open sets which covers E is reducible.* †

A set E with one of the properties A, B, C has them all and is called *bicompact* by Alexandroff and Urysohn.

Alexandroff and Urysohn have shown that the properties A_0, B_0, C_0 are the first of a series of sets of equivalent properties A_μ, B_μ, C_μ, obtained by replacing the word "enumerable" in the statement of Theorem I_0 by the phrase "of power μ at most." If μ is the power of the set E we obtain property A, etc.

This generalization may be regarded as involving a raising of the upper limit of the powers of the sets entering the conditions A_0, B_0, C_0. A different generalization is obtained by raising the lower limit. Thus property C_0 might be changed to read as follows.

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* Mathematische Annalen, vol. 92 (1924), p. 259. For a detailed discussion of this and related theorems see the article by Hildebrandt already cited. See also Tychonoff and Vedenissoff, Bulletin des Sciences Mathématiques, vol. 50 (1926), pp. 15–37.

† Because a finite covering of E results after a finite number of steps this statement of property C is equivalent to that usually given.

C_μ. *Every family of open sets of power μ which covers E is reducible.*

These considerations may be made still more general by limiting the range of powers above and below. For example if $\lambda \leq \mu \leq \nu$ are infinite cardinals we may consider the property

$C(\lambda \leq \mu \leq \nu)$. *Every family of open sets of power μ, $\lambda \leq \mu \leq \nu$, which covers E is reducible; therefore contains a subfamily which covers E and is of power less than λ.*

The property of Lindelöf, *Every infinite family of open sets which covers E contains an enumerable subfamily which also covers E,* [*] is a special case of the property $C(\lambda < \mu \leq \nu)$.

We proceed to extend the foregoing considerations to topological spaces in general. In this investigation properties analogous to A, B, C are fundamental. We first develop a series of propositions relative to coverings of an abstract type T.

15. A theorem on reducibility of coverings. Let μ be any cardinal number and let $\mathfrak{F} = [V_\alpha \mid \alpha < \Omega_\mu]$ be any infinite normal covering of E of type T and power μ. Denote the corresponding associated subset of E by $Q = [q_\alpha]$. Then $|Q| = \mu$, and the sequence

$$\mathfrak{S} = [G_\alpha], \quad G_\alpha = \sum_{\beta \geq \alpha} q_\beta,$$

is of power μ. Furthermore there is, by the definition of Q, no set V of $\mathfrak{F}$ in the relation T to q_β if $\beta > \alpha$. Consequently no set V_α can be in the relation T to a point of every set G_α of $\mathfrak{S}$.

The result just obtained is formulated in the following lemma.

LEMMA. *If T is a relation between a set V and an element of V, μ is a cardinal number, and $\mathfrak{F} = [V]$ is an irreducible covering of a set E of type T and power μ, then*

(A) *there is a subset of Q of E of power μ such that no set V of the family $\mathfrak{F}$ is in the relation T to μ elements of E.*

This implies further that

(B) *there is a decreasing sequence $\mathfrak{S} = [G]$ of subsets of E such that $|S| = \mu$ and no set V of $\mathfrak{F}$ is in the relation T to an element of every set G of $\mathfrak{S}$.*

From this lemma we readily conclude the following fundamental theorem:

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[*] Called the property of perfect separability by Fréchet. See Tychonoff and Vedenissoff, loc. cit., for a discussion of spaces (P, V) with this property.

THEOREM. *If μ is any cardinal number, T any relation between a set V and one of its elements, $\mathfrak{F} = [V]$ any covering of a set E of type T, the property*

(A) *there exists for every subset of Q of E of power μ a set V of $\mathfrak{F}$ in the relation T to μ points of Q,*
is implied by the property

(B) *there is for every decreasing sequence $\mathfrak{S} = [G]$ of subsets of E, such that $|\mathfrak{S}| = \mu$, a set of V of $\mathfrak{F}$ in the relation T to an element of every set G of $\mathfrak{S}$, and both properties imply*

(C) *the covering $\mathfrak{F}$ is either reducible or of power less than μ.*

16. Decreasing sequences of sets. The rôle of decreasing sequences of sets of points in the analysis of covering theorems was first exhibited for the classes (L) of Fréchet by R. L. Moore.* The symbol $\mathfrak{S} = [G]$ will be adopted as a fixed notation for a decreasing sequence of sets G of elements. The sequence $\mathfrak{S}$ is subject to the condition that given any two sets of $\mathfrak{S}$, one is a proper subset of the other. Suppose the sets G of $\mathfrak{S}$ are arranged in normal order. Evidently there will exist well ordered sequences in $\mathfrak{S}$ of the form

$$\mathfrak{S}_0 = [G_\alpha \mid \alpha < \Omega_\mu]$$

with the following property: every set G of $\mathfrak{S}$ contains all the elements of some set G_α of $\mathfrak{S}_0$. The sequences $\mathfrak{S}$ and $\mathfrak{S}_0$ are said to be *equivalent*.

Among all the well ordered sequences $\mathfrak{S}_0$ equivalent to a given sequence $\mathfrak{S}$ is a class of sequences of minimal cardinal number μ. This cardinal is regular, as the contrary hypothesis permits the existence of an equivalent sequence of lower cardinal number. The cardinal μ so determined will be called the regular cardinal belonging to the sequence $\mathfrak{S}$.

Let T be any relation between a set V and one of its elements. A sequence $\mathfrak{S} = [G]$ is *closed* in a set E with respect to a relation T if *there is a subset V of E which is in the relation T to an element of every set G of $\mathfrak{S}$.*

It is evident that if a sequence $\mathfrak{S}$ is closed the corresponding minimal sequences $\mathfrak{S}_0$ are also closed and vice versa.

If p is point of E common to every set G of a sequence $\mathfrak{S}$ and there is a set V in the relation T to p, then $\mathfrak{S}$ is closed in E by definition. For this reason we restrict the term *open sequence* to the case in which no element of E is common to all the sets G.

Let $\mathfrak{S} = [G]$ be an open sequence in a set E. There is a set $Q = [q_\alpha]$ of regular power μ such that for every $\alpha < \Omega_\mu$ there is a set G which does not

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* Proceedings of the National Academy of Sciences, vol. 5 (1919), pp. 206–210.

contain q_α. Furthermore there is no set V in the relation T to μ points of the set Q. Otherwise V would be in the relation T to an element of every set G of the sequence $\mathfrak{S}$. This proves the following theorem:

THEOREM. *If for every set Q of elements of a set E of regular cardinal number μ there is a subset of V of E in the relation T to μ elements of Q then every decreasing sequence $\mathfrak{S}$ of subsets of E whose regular cardinal is μ is closed in E.*

This theorem has the following corollary:

THEOREM. *If for every infinite set Q of elements of a set E there is a subset V of E in the relation T to $|Q|$ elements of Q, then*

(B) *every infinite decreasing sequence of subsets of E is closed in E with respect to the relation T.*

17. Decreasing sequences derived from normal coverings. Let $\mathfrak{F} = [V_\alpha | \alpha < \Omega_\mu]$ be an irreducible normal covering of a set E of type T and power μ. Let Ω_ν be the ordinal series of least cardinal number ν coterminal with Ω_μ. The regular cardinal ν is said to be the regular cardinal of the series Ω_μ, or of the cardinal μ. If μ' is a cardinal whose ordinal series $\Omega_{\mu'}$ is coterminal with Ω_μ it is easy to see that ν is the corresponding regular cardinal.

There is a subset $Q = [q_\beta | \beta < \Omega_\nu]$ of E such that q_β is in the relation T to V_{α_β} and not in that relation to V_α for any $\alpha < \alpha_\beta$. The indices α_β form a series coterminal with the series $\alpha < \Omega_\mu$. The cardinal number of Q is ν. Consider the decreasing sequence of subsets of E,

$$\mathfrak{S} = [G_\beta | \beta < \Omega_\nu], \quad G_\beta = \sum_{\gamma > \beta} q_\gamma.$$

Since no set V_α can contain an element of every set G_β we have the following theorem:

THEOREM. *If $\mathfrak{F}$ is an irreducible covering of a set E of type T and if ν is the corresponding regular cardinal there exists a decreasing sequence $\mathfrak{S} = [G]$ of subsets of E such that $|\mathfrak{S}| = \nu$, and no set V of $\mathfrak{F}$ is in the relation T to an element of every set G.*

The following two propositions are immediate consequences.

THEOREM. *If $\mathfrak{F}$ is an irreducible covering of a set E of type T, ν is a regular cardinal number, and if for every decreasing sequence $\mathfrak{S} = [G]$ of subsets of E such that $|\mathfrak{S}| = \nu$ there is a set V in $\mathfrak{F}$ which is in the relation T to an element of each set G of $\mathfrak{S}$, then ν is not the regular cardinal determined by the cardinal number of $\mathfrak{F}$.*

THEOREM. *If $\mathfrak{F}$ is any infinite covering of a set E of type T, and if for every infinite decreasing sequence $\mathfrak{S} = [G]$ of subsets of E there is a set V in $\mathfrak{F}$ which is in the relation T to an element of each set G of the sequence $\mathfrak{S}$, then $\mathfrak{F}$ is reducible to a finite covering.*

No set V of the family $\mathfrak{F}$ can contain ν elements of the set Q. For by the definition of q_β, V_α is not in the relation to q_{α_β} if $\alpha < \alpha_\beta$. Hence the elements of Q in the relation T to V_{α_β} form a subset of $Q - G_\beta$ which is of power less than ν because ν is a regular cardinal. This result is formulated in the following theorem.

THEOREM. *If $\mathfrak{F} = [V]$ is an irreducible covering of a set E of type T, ν is any regular cardinal number, and if for every subset Q of E of power ν there is a set V of $\mathfrak{F}$ in the relation T to ν elements of Q, then the cardinal ν is not the regular cardinal determined by the cardinal number of the family $\mathfrak{F}$.*

From this theorem we obtain as a corollary the following fundamental proposition:

THEOREM. *If $\mathfrak{F}$ is any covering of a set E of type T and if for every infinite subset Q of E there is a set V in the family $\mathfrak{F}$ which is in the relation T to $|Q|$ elements of Q then $\mathfrak{F}$ may be reduced to a finite covering.*

18. On the power of an irreducible covering. The following conditions are sufficient to ensure that the irreducible coverings of a given type T shall be of power less than a given infinite cardinal.

THEOREM. *Let $\mathfrak{F} = [V]$ be any irreducible covering of a set E of type T, and let λ be any infinite cardinal. Then if*

(A) *for every subset Q of E of power λ at least there is a set V of $\mathfrak{F}$ in the relation T to $|Q|$ points of Q,*
or if

(B) *for every decreasing sequence $\mathfrak{S} = [G]$ of subsets of E for which $|\mathfrak{S}| \geq \lambda$, there is a set V of $\mathfrak{F}$ in the relation T to an element of each set G of $\mathfrak{S}$,*
it follows that the power of $\mathfrak{F}$ is less than λ. Furthermore, property B *implies property* A.

The conclusion will read $|\mathfrak{F}| \leq \lambda$ if we replace the conditions $|Q| \geq \lambda$, $|\mathfrak{S}| \geq \lambda$, by $|Q| > \lambda$, $|\mathfrak{S}| > \lambda$.

The family $\mathfrak{F}$ may be assumed to be in the irreducible normal form

$$\mathfrak{F} = [V_\alpha],$$

where the index α ranges over the first well ordered series of cardinal number $|\mathfrak{F}|$. Let $Q = [q_\alpha]$ be the subset of E associated with $\mathfrak{F}$. Then $|Q| = |\mathfrak{F}|$.

If $|Q| \geq \lambda$ there is by (A) a set V_α of $\mathfrak{F}$ which is in the relation T to $|Q|$ points of Q. As this is contrary to the definition of Q as an associated set for the irreducible family $\mathfrak{F}$ it follows that the power of Q, and therefore of $\mathfrak{F}$, is less than λ.

To establish the second part of the theorem let

$$G_\alpha = \sum_{\beta > \alpha} q_\beta.$$

The sequence $\mathfrak{S} = [G_\alpha]$ cannot be of cardinal number λ or greater since there must exist in that case a set V_{α_0} of $\mathfrak{F}$ which is in the relation T to an element of every G_α, therefore to an element of G_β for $\beta < \alpha_0$. But q_β is by definition not in the relation T to any set V_α for $\alpha < \beta$.

That property (B) implies (A) can be shown quite easily by assuming that the elements of a set Q of power λ at least are arranged in a well ordered series of minimal ordinal type. From this series we obtain a sequence $\mathfrak{S}$ as in previous arguments. The set V of $\mathfrak{F}$ which contains a point of every set G of $\mathfrak{S}$ must contain $|Q|$ points of Q.

19. **Necessary and sufficient conditions that every infinite covering of type T be reducible.** In the preceding paragraphs we obtained sufficient conditions for the reducibility of coverings of power λ at least. The conditions become necessary if $\lambda = \aleph_0$.

THEOREM. *If E is any aggregate and $\mathfrak{F} = [V]$ is any normal covering of E of type T the following properties are equivalent:*

(A) *Every infinite subset A of E contains a subset B of power $|B| = |A|$ such that every element of B is in the relation T to a single set V of $\mathfrak{F}$.*

(B) *For every infinite decreasing sequence $\mathfrak{S} = [G]$ of subsets of E there is a set V of $\mathfrak{F}$ in the relation T to an element of each set G of $\mathfrak{S}$.*

(C) *The covering $\mathfrak{F}$ is finite.*

From the theorem of §15 we have at once B→A→C. It is therefore sufficient to prove that C→B.

Let $\mathfrak{S} = [G]$ be an infinite decreasing sequence of subsets of E and let $\mathfrak{F} = [V_1, V_2, \cdots, V_n]$ be a finite covering of E of type T. Let G_i denote a set of $\mathfrak{S}$ which has no element in the relation T to a set V_i of $\mathfrak{F}(i=1, 2, \cdots, n)$. Since the sets G_i form a finite monotone sequence they have a common element p. Since this element cannot be in the relation T to any set V_i of $\mathfrak{F}$ we have contradicted the hypothesis that $\mathfrak{F}$ covers E of type T.

20. **The theorem of Borel in general topological space.** If we consider the special case in which the relation T means "interior to" in the sense of the foregoing discussion, it is at once clear that the family $\mathfrak{F}$ of all possible

neighborhoods of the points of E is a covering of E of type T. In this case the theorem of the preceding section assumes the following form:

THEOREM. *If E is any set of points in a topological space and $\mathfrak{F}$ is any infinite proper covering of E the following properties are equivalent*:

(A) *Every infinite subset A of E contains a subset B of power $|B| = |A|$ such that B is interior to some set V of $\mathfrak{F}$.*

(B) *For every infinite decreasing sequence $\mathfrak{S} = [G]$ of subsets of E there is a set V of $\mathfrak{F}$ which has an element of every set G of $\mathfrak{S}$ in its interior.*

(C) *The covering $\mathfrak{F}$ is reducible.*

From this proposition we easily obtain the following generalization of the theorem of Borel.

THEOREM. *In a general topological space the following three properties of a set E are equivalent*:

(A) Sierpinski-Kuratowski. *Properly self-compact. Every infinite subset of E has a proper nuclear point in E.*

(B) R. L. Moore. *Perfectly properly self-compact. Every infinite decreasing sequence of subsets of E is closed in E.*

(C) Borel. *Bicompact. Every infinite proper covering of E is reducible.*

With this proposition the second theorem of Alexandroff and Urysohn cited in §14 above receives a final generalization.

21. **Perfectly compact sets in a topological space.** We have shown that the generalization of the theorem of Borel to topological space leads to the introduction of the property *perfectly properly self-compact* instead of the property perfectly self-compact. The two properties coincide in any V-space in which derived sets are closed, in particular in a Hausdorff space. The following results are presented for comparison with those of the preceding section.

THEOREM. *Every perfectly compact set in a topological space is compact.*

Let E be a perfectly compact set in a topological space and let A be any infinite subset of E. Assume the elements of A arranged in an irreducible ordinal series:

(1) $$p_0, p_1, \cdots, p_\alpha, \cdots \qquad (\alpha < \Omega),$$

and let

$$G_\alpha = \sum_{\beta \geq \alpha} p_\beta.$$

The sequence $\mathfrak{S} = [G_\alpha | \alpha < \Omega]$ is decreasing and $\Pi G_\alpha = 0$. By hypothesis the

sequence $\mathfrak{S}$ is closed in the space (P, K), that is, there is a point p common to the sets $K(G_\alpha)$, $\alpha < \Omega$. Since $G_0 = A$, p belongs to $K(A)$. Hence E is compact.

THEOREM. *If a set E in a topological space is perfectly compact every infinite subset of E of regular cardinal number has a nuclear point.*

Let H be a subset of E of regular cardinal number and let Ω be the least ordinal corresponding to the cardinal $|H|$. Let a series of the type (1) above denote a representation of H as an ordinal series. The corresponding decreasing sequence $\mathfrak{S}$ is closed in E, that is, there is a point q of E which is common to all the sets $K(G_\alpha)$.

Let V be any neighborhood of the point q. There is an ordinal α_0 such that for all $\alpha > \alpha_0$, G_α does not contain q. For each $\alpha > \alpha_0$ there is a point q_α of G_α in V. Let Q be the set of all distinct points q_α so defined. For a given element q_α of Q let β denote the index such that q_α is in G_β but not in $G_{\beta+1}$. The index β is in fact that index α which corresponds to q_α regarded as an element of H and determined by the correspondence (1). These indices β are such that for every α there is a $\beta > \alpha$. Since the cardinal $|H|$ is regular it follows that $|Q| = |H|$. Therefore the point q is a nuclear point of the set H.

Sierpinski has shown by an example for the case of spaces (P, V) that this theorem cannot be extended to sets of irregular cardinal number.*

THEOREM. *If every infinite subset H of regular cardinal number of a set E in a topological space has a nuclear point and $\mathfrak{S} = [G]$ is an open decreasing sequence of subsets of E, then there is a point common to the sets $L(G)$.*

This theorem can be deduced from the results of §16. It is sufficient to state that a point p is in the relation T to a set V if V contains p.

Since every space (P, L) is a space (P, V) it will be seen that we have obtained a generalization of a theorem which I demonstrated in an earlier article for sets of regular cardinal number.†

THEOREM. *A necessary and sufficient condition that a set E in a space (P, L) derived from a topological space be perfectly compact is that every infinite subset of E of regular cardinal number have a nuclear point.*

The following example of a non-monotonic set-function $K(E)$ shows that this theorem admits no further generalization.

22. Example of a space (P, K) which is nuclear but not perfectly compact. Let P be the class of all positive integers. Let

$$(1) \qquad\qquad S_1 = p_1, \ p_2, \ p_3, \ \cdots$$

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* Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 649–653.

† Bulletin of the American Mathematical Society, vol. 30 (1924), p. 511.

be the sequence of all prime numbers. Let

(2) $S_n = p_n p_{n+1}, \; p_{n+1} p_{n+2}, \; p_n p_{n+3}, \; \cdots .$

Let

$$G_n = S_n + S_{n+1} + S_{n+2} + \cdots .$$

The function $K(E)$ is defined as follows. If E is finite, $K(E) = 0$. If E is infinite and n is the least integer for which E contains all but a finite number of elements of S_n, $K(E) = p_n$. If E is infinite and does not satisfy this condition, $K(E) = 1$.

The sequence $G_1, G_2, G_3, \cdots$ is decreasing and $K(G_n) = p_n$. Thus the space is not perfectly compact, although it is perfectly L-compact.

The space is nuclear. Let q be a K-point of a set E, and let Q be any set containing only a finite number of points of E. Then by definition, q is a K-point of $E - EQ$ and cannot be interior to Q. Thus Q is not a neighborhood of the element q. The function $K(E)$ is obviously not monotonic. This example may be extended to include the case of nuclearity of any order.

23. Perfectly self-compact sets in a topological space. The following theorem generalizes a result of Sierpinski.* The method of proof was suggested by W. L. Ayres.

THEOREM. *Every perfectly self-compact set E contains a nuclear point of itself.*

Suppose the theorem is not true. Then for every point p of E there exists a neighborhood V_p such that

$$| V_p \cdot E | < | E | .$$

The family $\mathfrak{F}$ of neighborhoods V_p so defined is a proper covering of E. Let

$$\mathfrak{F}_1 = [V_0, V_1, \cdots, V_\alpha, \cdots, | \alpha < \Omega_\mu],$$

where $\mu \leqq |E|$, be the corresponding normal subfamily, and let $Q = [q_\alpha]$ be the corresponding associated set.

Consider the decreasing sequence

$$G_0 = E, G_1 = E - EV_0, \cdots, G_\alpha = E - E \cdot \left(\sum_{\beta < \alpha} V_\beta \right).$$

Since E is perfectly self-compact, there exists an element p of E which, for every α, belongs to $K(G_\alpha)$. But p is interior to some set V_{α_0} of $\mathfrak{F}_1$. And if $\beta_0 > \alpha_0$, then $G_{\beta_0} = E - E \cdot (\sum_{\beta < \beta_0} V_\beta)$ has no point in common with V_{α_0}. Hence p does not belong to $K(G_{\beta_0})$, the desired contradiction.

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* Bulletin of the American Mathematical Society, loc. cit.

III. Properties of the class of all continuous functions on a topological space

24. Continuous functions on a topological space. Let $M(f, V)$, $m(f, V)$ denote the upper and lower bounds of a real valued single valued finite function $f=f(p)$ on a set V. Then

$$w(f,V) = M(f,V)$$

is the oscillation of the function f on the set V if these bounds are finite. If one or both of the bounds are infinite the oscillation is infinite.

If the oscillation of a function f has the minimum zero on the family of all possible neighborhoods of a point p, the function is continuous at p.* A function is continuous on a set E if it is continuous at every point of E. Continuity relative to a set E is defined by means of the oscillation on the set EV.

Denote by A_f the set of all functional values of a function $f(p)$. Let M, m (finite or infinite) be the upper and lower bounds of A_f.

The notation $E(f<a)$ refers to the set of all points at which the value of the function f is less than a. The meaning of the notations $E(f\leq a)$, etc., is evident.

For every continuous function $f(p)$ on a topological space, the sets $E(f<a)$, $E(f>a)$ are *open*, and the sets $E(f\leq a)$, $E(f\geq a)$ are completely closed.

25. Continuous functions on a neighborhood space. It is important to observe that the preceding definition of continuous function is a neighborhood definition. It follows readily that if.(P, V) is the neighborhood space or class (V) derived from a topological space (P, K), then the class of all continuous functions is the same for both spaces. This implies that under this definition the theory of continuous functions attains its greatest generality in the classes (V).

26. Linear series of closed sets. Let $f(p)$ be any function on a class P. The set $L_x=E(f=x)$, of all points p such that $f(p)=x$, will be called the x-level of the function f. If $x\neq x'$ then $L_x L_{x'}=0$. If the function f is continuous, the set L_x is completely closed. Furthermore, if we set $m_f=a$, $M_f=b$, we obtain for each value of x in the interval $a\leq x\leq b$ a set L_x (which may be null) and the set

$$F_x = \sum_a^x L_x$$

is completely closed for each value of x. A series of closed sets of this type will be called a linear series.

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* Fréchet II, p. 363.

A necessary and sufficient condition that a linear series $L_x (a \leq x \leq b)$ correspond to a continuous function $f(p)$ on a topological space (P, K) is that the set

$$\sum_{x=x'}^{x=x''} L_x$$

be completely closed for every pair of values of x', x''.

27. **Conditions for the existence of non-constant continuous functions.** It is evident that the theory of continuous functions on a space is barren in case all continuous functions are constant. This is true, for example, for all enumerable connected spaces. Fréchet has proposed, in correspondence with Urysohn and myself, the following problem: *characterize those spaces in which there exists a non-constant continuous function.* This problem has been solved for the spaces of Hausdorff by Urysohn.† His solution is easily extended to topological spaces in general.

A family of open sets G is *normal* provided

(a) there exist two sets G_0, G_1 of the family which are distinct and such that $G_0 < G_1$;

(b) if G_0, G_1 are any two sets of the family such that $G_0^* \leq G_1$,‡ there is a set G such that

$$G_0^* \leq G \leq G^* \leq G_1.$$

THEOREM (Urysohn). *A necessary and sufficient condition that a topological space admit the existence of a non-constant continuous function is that it contain a normal family of open sets.*

As the proof of this proposition given by Urysohn for Hausdorff spaces can be extended without serious changes to the general case the demonstration is omitted.

It should be noted that the requirement of normality is significant only in connected spaces. The preceding theorem has the following corollary:

THEOREM (Urysohn). *Every connected topological space which contains a normal family of open sets has the cardinal number of the continuum at least.*§

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† *Über die Mächtigkeit der zusammenhängenden Mengen*, Mathematische Annalen, vol. 94 (1925), p. 290.

‡ The set G^* is the least completely closed set containing G. See §8 above.

§ Urysohn (loc. cit.) has given an example of an enumerable connected set on which every continuous function is constant. The following space illustrates this possibility very simply. Let P be the class of all positive integers. For each integer p the mth neighborhood $V_{p, m}$ is the set $[p; m+1, m+2, \cdots]$.

28. **Another form of the problem of the existence of non-constant continuous functions.** Fréchet has proposed the problem of the existence of non-constant continuous functions in the following form: *characterize the spaces such that for every subset E there is a continuous function on the space which is not constant on E.*

In this case it is both necessary and sufficient that there exist for every two distinct points p, q a function f, continuous on P, such that $f(p) = f(q)$. And as in the preceding article it is necessary and sufficient that there exist for each pair of points p, q a normal family of open sets G, and in this family two sets G_0, G_1, such that

$$p < G_0 < G_1$$

and q is not an element of G_1*.

29. **The neighborhood space defined by the class of all continuous functions on a topological space.** In terms of the class C of all continuous functions on a topological space we define a neighborhood space (P, W) in which the set-function W is defined by the family of all possible sets

$$W = E(0 \leq g < 1/n),$$

where g denotes a non-negative continuous function.

The space (P, W) is easily seen to be monotonic, accessible and regular.† It has the properties A and B of Hausdorff. It is at once evident that the set-functions V and W derived from a postulated set-function K will not be equivalent in general. Since every neighborhood W is a neighborhood V, the condition for the equivalence of the two systems of neighborhoods reduces to the following: every neighborhood V contains a neighborhood W. Hence

A necessary and sufficient condition that the set-functions $W(E)$, $V(E)$ derived from a set-function $K(E)$ be equivalent is that there exist for every point p and neighborhood V of p a non-negative continuous function g such that for some value of n the set $W = E(g < 1/n)$ contains p and is a subset of V.

The following theorem is equally evident.

THEOREM. *The space (P, K) is equivalent to its derived space (P, W) with respect to continuity. That is, every function which is continuous on one space is continuous on the other.*

30. **Analysis of the properties of the class of all continuous functions.** Certain properties of the class C of all continuous functions on a topological

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† A space is regular provided there exist, for every point p and completely closed set B not containing p, a pair of disjoined open sets $U > p$, $V > B$. Since p is not a W-point of B there is a function g and an integer n such that the set $U = [0 \leq g < 1/n]$ contains p and no point of B, while $V = [g > 1/n]$ contains B. This does not imply that every pair of points is separated by open sets.

space (P, K) hold for all topological spaces. Among these properties are the following:

The class C contains all constant functions. If a function f belongs to C and g is a constant, f+g belongs to C, likewise fg.

If f is a continuous function of a real variable x and g belongs to C, then f(g) belongs to C.

The following simple example shows that the sum or product of two continuous functions may be discontinuous. Let $P = [a, b, c]$ be a space of three elements. The only point of accumulation is a which is a K-point of $E = [b, c]$ only. Let $f(a) = f(b) = 1$, $f(c) = 2$; $g(a) = g(c) = 3$, $g(b) = 4$. Then f and g are continuous on P while their sum and product are both discontinuous.

It is quite easy to show that if the class C is additive it is multiplicative.

THEOREM. *If C is multiplicative it is additive.*

Suppose C is not additive at a point p. Then functions f, g exist such that

$$h = f + g$$

is discontinuous at p. We may assume that $|f| \leq 1$ and that $g(p) = 0$. Since for every e there is a neighborhood U of p on which osc $g < e$ and therefore $|g| < e$ we have, at once, fg vanishes and is continuous at p. But the function

$$(1 + f)(1 + g) = 1 + f + g + fg$$

is certainly the discontinuous product of continuous functions.

Thus the algebraic closure properties of C are of two kinds; those that condition the space and those that do not. The fundamental property of the first kind is the additive property:

(1) *The sum of any two functions of the class C is a function of class C.*

It will be noted that the statement of the property is in no way related to the property of continuity assumed for the functions of the class C. In attempting to characterize a class of functions which is the class C this property would naturally be assumed in spite of the restriction on the space (P, K) which it involves. The problem of characterizing the class C varies with the space P. For example on a space of one element all functions are continuous. In the space of the previous example the class C is characterized by the fact that there is a point $p(=a)$ such that every function of f which has the value $f(p)$ at some other point of P is a function of C. It is evident that both of these characterizations involve properties not normally possessed by the class of all continuous functions.

Many of the classic properties of the class C of all continuous functions on a closed finite linear interval are in fact properties of the set A of functional

values. Thus A is itself a closed finite interval. This statement is implied by the statement that a continuous function f is bounded, attains its bounds, and every value between. It will be shown in §§32, 33 that these three properties are consequences of the following two:

(2) *For every function f of the class C the set A_f is closed.*

(3) *For every function f of the class C the set A_f is dense on an interval.*

The separation of the points of space is implied by the property

(4) *If p and q are any two distinct points there is a function f of C such that* $f(p) \neq f(q)$.

The class C should possess the fundamental property of Weierstrass:

(5) *There exists in C a finite or enumerably infinite family G of linearly independent functions*

$$g_1, g_2, \cdots, g_n, \cdots$$

such that each function f of C is the limit of a uniformly convergent sequence of functions of G.

It will be shown that the five properties so far stated are independent. The question of their complete independence has not been attacked.

The property of uniform continuity is easily defined for metric spaces in terms of the distance between two points. Since the distance $f(p) = \delta(p, q)$ between a fixed point p and a variable point q is a continuous function of p it is evident that uniform continuity is essentially a relationship between the class C and a subclass of C. The property of uniform continuity thus takes the following form:

(6) *There exists in C a family of functions $u(p)$, with the following properties:*

(a) *if p is a point there is a unique function u_p associated with p;*

(b) *if f is a function of C and e is a positive number there is a number* $d(>0)$ *such that for each point p and corresponding function u,* $|f(p) - f(q)| < e$ *whenever* $|u_p(p) - u_p(q)| < d.$

If this property is present the functions of C will be said to be uniformly continuous relative to the family $[u]$. The relation between this property and its predecessors has not been exactly determined. It is however quite easy to see that a class C with this property is also (1) additive.

In §29 we called attention to the derived space (P, W) defined by the class C and found the condition that this space be equivalent to the space (P, V) derived from a postulated space (P, K). This condition is the seventh of our list of fundamental properties. It is the first to utilize explicitly the given limiting relations.

(7) *For every point p and neighborhood V of p such that $C(V) = P - V$ is not null, there is a function g of C which is bounded from $g(p)$ on $C(V)$.*

Here the function g is dependent on the neighborhood V. In case it is uniformly independent of V we obtain a property denoted by (7').

Since the property (7) implies the equivalence of the spaces (P, K) and (P, W), all properties of the latter space become properties of the space (P, V) in the presence of (7).

The property of points expressed in (7) provides a basis of classification. A point for which this condition fails involves an interesting singularity of the function $K(E)$.

If the class C has this property (7) the space (P, V) must be regular and accessible.

It is easily shown by extending the proof of a theorem of Urysohn* to topological spaces in general that if the space is normal† the following property is present:

(8) *If A and B are any two completely closed disjoined sets there exists a function f of C, $0 \leq f \leq 1$, such that $f = 0$ on A, $f = 1$ on B.*

The following sections are devoted to a detailed study of the relationships between these eight fundamental properties of the class C and the corresponding properties of the underlying space (P, K). The following relationships are quite easily established. The first property is implied by the sixth and seventh respectively, the seventh is a consequence of the eighth.

31. Spaces for which the class C is additive. If the class C is additive it must be additive at each point of space. If a point p is such that every continuous function is constant on some neighborhood V of p, the class C is additive at P. Points of this character will be called level points of space. A *level L* of a space P is a set such that every continuous function is constant on L. The levels of a space P are completely closed sets.‡

If a point p is not a *level* point, there is for every neighborhood V of p a continuous function f which is not constant on V. From the result of §27 we see that the function f determines a normal family of open sets V each containing p. A point which is not a level point will therefore be called a *normal* point. It is evidently a regular point.

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* Loc. cit. See also E. W. Chittenden, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 17–18.

† A space is normal in case for every two completely closed disjoined sets A, B, there exist disjoined open sets V, U such that $A \leq V$, $B \leq U$.

‡ The following example shows that a level need not be connected. Let P be the set of all positive integers and let the points 1, 2 be points of accumulation of every infinite subset of P. Then the disconnected set 1, 2 is a level of P. It is, however, necessary that P be connected relative to a level L. That is, if $P = A + B$ where $H(A, B) = 0$, then one of the sets A, B must contain L. For any point p the associated level L is the set common to all the sets $E(f = f(p))$ for a continuous f.

THEOREM. *The points of a topological space fall into the two classes, level points and normal points.*

A point p at which the class C is not additive is normal. There must exist continuous functions f, g, such that their sum $f+g$ is discontinuous at p. It follows from the definition of continuity that the minimum of the oscillation of $f+g$ on the neighborhoods of p is positive. Since open neighborhoods V, W of p exist on which the oscillations of f and g respectively are arbitrarily small it follows that in general the product VW will not be a neighborhood of p. The following theorem can be easily inferred.

THEOREM. *A necessary and sufficient condition that the class C be additive at a normal point p is that there exist two normal families of open sets containing p such that if U is a set of one family and W a set of the other, then the set UW is an open set.*

The following theorem is now evident.

THEOREM. *In a topological space satisfying the second axiom of Hausdorff, that is, the condition that the product of open sets is an open set, the class C is additive.*

In particular, if the neighborhoods of each point can be defined in terms of an écart $\delta(p, q)$, this condition is satisfied. From §36 we see that property (6) implies property (1).

THEOREM. *If the class C is* (1) *additive the associated derived space (P, W) has the second property of Hausdorff, the function $W(E)$ has the fundamental properties* I, II, III, *and the space (P, W) is both accessible and regular.*

32. **Spaces for which every continuous function is bounded.** It is easy to see that if every continuous function is bounded, every continuous function attains its bounds. For if a real number a is a bound of a continuous function f and is not attained it must be a limit of functional values. Hence the function $1/(f-a)$ is both continuous and unbounded. Likewise the set A_f of functional values of a continuous function f must be closed. Conversely, if the set A_f is closed for every continuous function f, every continuous function must be bounded. Otherwise for some continuous function f, the continuous function $1/(1+|f|)$ would fail to attain its lower bound zero.

THEOREM. *A necessary and sufficient condition that every continuous function on a topological space be bounded is* (2): *the set of functional values of every continuous function is a closed set.*

If a topological space admits an unbounded continuous function f there exists a corresponding normal series of open sets (coterminal with a series of

type ω) which have no common element. Conversely the existence of a normal series of this type permits the definition, by an obvious modification of the method of Urysohn referred to in §27, of an unbounded continuous function.

THEOREM. *A necessary and sufficient condition that for every function f of the class C of all continuous functions the set A_f be limited or closed is that every normal series of open sets which is coterminal with a series of type ω determine at least one point common to the sets of the series.*

If the space (P, V) derived from a topological space is compact every continuous function is bounded. But there exist non-compact spaces on which every continuous function is bounded, for example, in case a fixed point p_0 is an element of every neighborhood of every point.

33. **Spaces for which the set of functional values of every continuous function is dense on an interval.** Suppose that the set A_f is dense on an interval I for some continuous function f, but that A_f is not a continuum. Then $A_f = A_1 + A_2$, where neither of the sets A_1 or A_2 contains a point or limit point of the other. Let P_1, P_2 be the subsets of P on which the functional values of f lie in A_1, A_2 respectively. The function equal to 0 on p_1, 1 on P_2 is continuous, and its set of functional values is not dense on an interval.

THEOREM. *A necessary and sufficient condition that the set A_f be dense on an interval for every continuous function (f) is that the space be connected. Under either hypothesis every continuous function attains all values between its bounds.*

34. **The property (4).** In this section we consider spaces for which the class C has the property

(4) *If p and q are distinct points there is a function f of C such that $f(p) \neq f(q)$.**

THEOREM. *A necessary and sufficient condition that the class C on a topological space have the property (4) is that for every pair of points p, q there is a normal series of open sets containing p of which q is not a common element.*

If a single function is effective for all pairs of points the space is homeomorphic with a subset of the linear continuum. The property (4) is satisfied vacuously if there is but one point in space. It is present in any space which is disconnected between every pair of points.

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* Because of the close relationship between properties (4) and (7) the following example is of interest. Let $P = [p + \{p_n\} + \{\{p_{mn}\}\}]$. Let p be the sequential limit of p_n, and p_n the sequential limit of p_{mn}. There are no other limit relations in P. No subset of the set $E = \{\{p_{nm}\}\}$ has p for a point of accumulation. Hence the set $V = [p + \{p_n\}]$ is a neighborhood of p. But the values assumed on E by any continuous function on P must have $f(p)$ for a limiting value. In this case the class C has property (4) but not property (7).

The Hausdorff property (D) (for every two points p, q there exist disjoined neighborhoods V_p, V_q) is implied by property (4). The third and fourth conditions of F. Riesz, a finite set has no point of accumulation, and a point is uniquely determined by the sets of which it is a point of accumulation, are satisfied in the derived space (P, V) when property (4) is present.

If the function f of property (4) is independent of the point q we obtain a property (4′) which appears to be topologically independent of (4), although the exact nature of the difference is not evident. If the space is compact and has property (4′), it is easy to show that it has the properties (A), (B), (D) of Hausdorff and also satisfies the first axiom of enumerability.

35. The fifth property. This property is suggested by the Weierstrassian theorem that a continuous function of a real variable is the limit of a uniformly convergent sequence of polynomials. This property is present in every space which can be represented as the sum of a finite number of distinct levels.

Let C_0 be the family of functions

$$g_1, \ g_2, \ g_3, \ \cdots .$$

Let b_n be the lower bound of g_n and set

$$Q_{nk} = E[b_n \leqq |g_n| < b_n + 1/k].$$

The family of all sets Q_{nk} obtained by varying the integers n and k is enumerable and equivalent to the family of neighborhoods W determined by the class C. In fact, if g is any continuous function, and $W = E[|g - g(p)| < a]$, we may, because of the uniform convergence of a sequence of functions g_{n_m} of C_0 to the continuous function $f = g - g(p)$, find an integer m for which $|g_{n_m} - f| < a/2$. Consequently the points of the set $Q_{n_m k}$ on which $|g_{n_m}| < a/2$ belong to the set E. Thus every neighborhood W contains a set Q_{n_m}. The sets Q_{n_m} are by definition open sets. It follows that the space (P, W) is perfectly separable.

THEOREM. *If the class C has the property* (5) *the corresponding space* (P, W) *is perfectly separable.*

That the Weierstrassian property (5) is present in any separable metric space follows readily from the theorem of Urysohn that every such space is homeomorphic to a subset of the Hilbert space.*

36. On uniform continuity relative to a family of continuous functions. The sixth property. In the sixth fundamental property of the class C we assume the existence of a family of continuous functions u_p, one for each point

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* Mathematische Annalen, vol. 92 (1924), p. 302.

of space, such that for every continuous function f and positive number e there is a number d such that if $|u_p(p) - u_p(q)| < d$ then $|f(p) - f(q)| < e$. It will be convenient to introduce the distance function

$$\delta(p, q) = |u_p(p) - u_p(q)|$$

which is continuous in q for fixed values of p.* The sixth property is possessed by all spaces which are composed entirely of isolated elements, since in this case all functions are continuous, and also uniformly continuous relative to an arbitrary class of functions u_p. If a space of this type has infinitely many points, we obtain unbounded uniformly continuous functions.

The space (P, δ) defined by this distance function $\delta(p, q)$ is equivalent to the space (P, W) determined by the entire class C. For since u_p is a continuous function its oscillations on the neighborhoods W of p have zero for their minimum. Likewise the oscillation of a continuous function f on the neighborhood $V_n = [\delta(p, q) < 1/n]$ has the minimum zero because of uniform continuity.

THEOREM. *If the class C has the property* (6) *it is* (1) *additive and there is an écart* $\delta(p, q)$, *continuous in* q, *such that the spaces* (P, δ) *and* (P, W) *are equivalent.*

37. **The seventh property.** Consider the property (7): for every point p and neighborhood V of p such that the complementary set $C(V) = P - V$ is not null, there is a continuous function f which vanishes at p and is bounded from zero on $C(V)$.

It is shown in §29 that this property is the condition that the spaces (P, V) and (P, W), where the set function W is defined by the class C, be equivalent. Thus a space (P, V) in which (7) is satisfied is required to be both accessible and regular.

A necessary and sufficient condition that a space with the property (7) be a regular Hausdorff space is that it satisfy the first axiom of separation, that is, that every pair of distinct points lie in disjoined open sets, and that the class C be additive.

In particular if the class C has the properties (1), (4), (7), the space (P, V) is a regular Hausdorff space.

From properties (6), (7), we may conclude that the space (P, δ) of §36

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* In general the function $\delta(p, q)$ is unsymmetric. It does not appear to have been previously observed that the assumption of a *uniformly continuous* symmetric écart $\delta(p, q)$ is essentially equivalent to the assumption of the axiom of the triangle, $\delta(p, q) \leq \delta(p, r) + \delta(r, q)$. Note added during correction of proof: This proposition was established by Menger, Mathematische Annalen, vol. 100, Nos. 1-2, August 28, 1928, p. 145.

is equivalent to the space (P, W) of §29. The results of this section are summarized in the following theorem.

THEOREM. *If in a space (P, V) the class C of all continuous functions has the property* (7), *the space is accessible and regular. If properties* (1), (4) *are present the space is regular Hausdorff space. Properties* (5), (7) *together ensure that the space be accessible, regular and perfectly separable (therefore normal). Properties* (6) *and* (7) *together imply that the space (P, W) is equivalent to a space (P, δ) where $\delta(p, q)$ is continuous in q.*

38. A stronger property. We consider some consequences of the presence of the property (7′) obtained by making the function g of (7) independent of the neighborhoods of p.

(7′) *For any point p there is a function g which is bounded from $g(p)$ on $C(V)$ for every neighborhood V of p whose complement is non-null.*

Since (7′) implies (7) it follows at once that the space is regular and accessible. But it is quite easy to see that the product of any two open sets is an open set.

This property permits a statement of the converse of the proposition, every continuous function on a compact space is bounded. It is possible to show that if a space (P, V) with the property (7′) is not compact there exists an unbounded continuous function.

Since the space is not compact there exists an infinite set A of points with no point of accumulation. For any point p the set $V = P - A - p$ is a neighborhood, and $C(V) = A - p$ is not null. Then by hypothesis there is a function which vanishes at p and at no other point. Let $Q = [q_n]$ be an enumerable set which has no V-point. Then for each point q_n we have a continuous function g_n, $0 \leqq g_n(q_n) \leqq 1$, which vanishes at q_n and at no other point. Let us choose a closed neighborhood $\bar{V}_m$ which contains no point of $Q - q_m$. On $C(\bar{V}_m)$, $g_m \geqq d_m > 0$. It is possible by a well known process to define integers k_m so that the infinite product

$$ f_n = \prod_{m=n} g_m e^{\gamma_m}, $$

where

$$ \gamma_m = 1 - g_m + (1 - g_m)^2/2 + \cdots + (1 - g_m)^{k_m}/k_m, $$

converges uniformly in some neighborhood of every point p. It is sufficient to choose k_m equal to the larger of $1/d_m$ and $m \lvert (\log 2)/\log (1 - d_m) \rvert$. The function f_n so defined is continuous, vanishes at q_n, q_{n+1}, $\cdots$, and at no other points. The function $h_m = f_n$, if $f_n \leqq 1$, and $= 1$, if $f_n = 1$, is continuous. The function

$$ h = \sum_{1}^{\infty} \frac{1}{2^n} h_n $$

is continuous, does not vanish, and satisfies the inequality

$$h(q_n) = \frac{1}{2^{n+1}} h_{n+1}(q_n) + \frac{1}{2^{n+2}} h_{n+2}(q_n) + \cdots \leqq \frac{1}{2^n}.$$

Therefore the continuous function $1/h$ is unbounded.

THEOREM. *In a space with property* (7') *a necessary and sufficient condition that* (2) *every continuous function be bounded, is that the space be compact.*

It would be of interest in this connection to determine the most general topological space in which compactness is equivalent to property (2).

THEOREM. *If the class C of all continuous functions on a neighborhood space* (P, V) *has the property* (7'), *then the space is regular and satisfies the Hausdorff axioms* (A), (B), (C). *The class C is additive. If every continuous function is bounded the space is compact.*

39. **The property of Hahn.** The eighth of the series of properties discussed may be called the property of Hahn because it appears as part of Hahn's proof of the existence of non-constant continuous functions in a space admitting a uniformly regular écart (originally called "voisinage" by Fréchet). The class C on a space (P, V) has the property (8) *if for every pair of disjoined closed sets, A, B, there exists a continuous function f,* $0 \leqq f \leqq 1$, *which vanishes on A and is equal to* 1 *on B.*

This property stated for (P, V) is equivalent to the property defined for spaces (P, K) by replacing the word closed by "completely closed." Properties (7) and (7') are immediate consequences of (8). The importance of this property can be inferred from the following theorem which is essentially due to A. D. Pitcher:

THEOREM (Pitcher). *If the class C has the properties* (2), (4), (6), (8), *the space* (P, K) *is a compact (separable) metric space. The class C has the additional properties* (1), (2), (5), (7), (7').

This theorem follows from the fact that (8) implies (7), the preceding theorem, a theorem of A. D. Pitcher,* the equivalence of "distance" and "regular écart," the well known fact that every compact metric space is separable,† and the fact that all the properties mentioned are present if P is a metric space.

40. **Independence.** The following list of examples shows that the properties (1)-(5) of the class C are independent. That property (7) is indepen-

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* A. D. Pitcher, loc. cit., p. 76, Theorem 19.
† Fréchet, Palermo Rendiconti, vol. 30 (1910), p. 1–26.

dent of (1)-(6) is also established. Similarly (7′) is independent of (1)-(7). We have no example showing that (8) is independent of the remaining eight properties, and there are reasons for thinking it is not. While properties (2) and (6) are independent, as is shown by examples (5) and (8), we have no example showing that (6) is independent of (1)-(5). It would be interesting to determine the exact relationship between boundedness and uniform continuity.

In the following examples the properties not present are indicated. The proofs will be left to the reader.

1. (1), (6), (7), (8). Let $P = [0 \leq p \leq 1; 2]$. Limit is defined as usual for the points of the interval, $0 \leq p \leq 1$. Let $p_{2n} = 1/n$, $p_{2n+1} = 1 - 1/n (n = 1, 2, 3, \cdots)$. The neighborhoods of the point 2 are $V_{2n} = [2; p_{2n}, p_{2n+1}, \cdots]$, $V_{2n+1} = [2; p_{2n+1}, p_{2n+1}, \cdots]$.

2. (2), (6). The class P is the segment $0 < x < 1$, with limit defined as usual.

We have no example of a space with all of the properties except (2).

3. (3). P is composed of two isolated points.

4. (4). P is the interval $0 \leq p \leq 1$, together with the point $p = 2$. Limit is as usual, except that the point $p = 2$ is in every neighborhood of $p = 0$.

5. (5), (6). The class P is the space defined by Hausdorff* composed of a non-enumerable well ordered set with the interstices filled by linear continua.

We have no examples of the class (5) or (6).†

6. (7), (7′), (8). The class P is the interval $0 \leq p \leq 1$, limit is defined as usual except that the point $p = 0$ has its neighborhoods defined by the formulas [all irrational numbers $p < 1/n$], [all rational numbers $p < 1/n$] $(n = 1, 2, 3, \cdots)$.

7. (7′), (8). Let P be the interval $0 \leq p \leq 1$. The neighborhoods of all points $p > 0$ are as usual. From each neighborhood of zero remove its middle point. Then the space P is not compact, for the sequence of points $1, 1/2, 1/3, \cdots$ has no point of accumulation. The class C of all continuous functions is unaffected by this modification of the neighborhoods of zero.

8. (2), (3). The class P is the class of positive integers, with each point isolated.

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* Tietze, Mathematische Annalen, vol. 91 (1924), p. 218, Example B₈.
† Examples 2 and 5 show the independence of properties (2) and (6).

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